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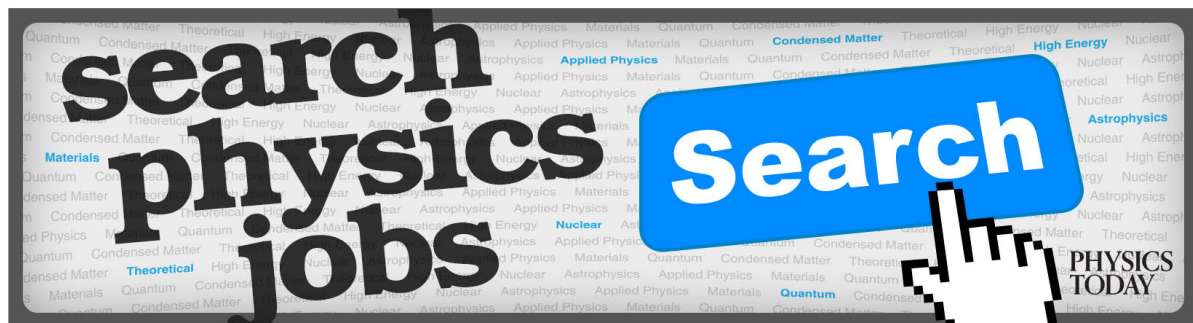
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Non-Markovian quantum feedback networks II: Controlled flows

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The concept of a controlled flow of a dynamical system, especially when the controlling process feeds information back about the system, is of central importance in control engineering. In this paper, we build on the ideas presented by Bouten and van Handel [*Quantum Stochastics and Information: Statistics, Filtering and Control* (World Scientific, 2008)] and develop a general theory of quantum feedback. We elucidate the relationship between the controlling processes, Z , and the measured processes, Y , and to this end we make a distinction between what we call the input picture and the output picture. We should note that the input-output relations for the noise fields have additional terms not present in the standard theory but that the relationship between the control processes and measured processes themselves is internally consistent—we do this for the two main cases of quadrature measurement and photon-counting measurement. The theory is general enough to include a modulating filter which post-processes the measurement readout Y before returning to the system. This opens up the prospect of applying very general engineering feedback control techniques to open quantum systems in a systematic manner, and we consider a number of specific modulating filter problems. Finally, we give a brief argument as to why most of the rules for making instantaneous feedback connections [J. Gough and M. R. James, *Commun. Math. Phys.* **287**, 1109 (2009)] ought to apply for controlled dynamical networks as well. *Published by AIP Publishing.* [<http://dx.doi.org/10.1063/1.4989992>]

I. INTRODUCTION

Markovianity is one of the most frequently discussed and misunderstood ideas in physics. The essential problem is the concept of the *state* of a system, and this goes back to Hamilton who took Newton's second order differential equations of motion and converted them into first order equations for the mechanical state (the positions and momenta). The subsequent development of mechanics by Hamilton, Jacobi, Liouville, and Poisson showed that the collection of states (phase space) was more than just a blank canvas but had its own canonical structure. At its heart, Hamiltonian mechanics is as far reaching as it is because it tells us how to define the appropriate state and how to propagate it forward in time. Markov effectively extended this to stochastic systems by once again having a state space for the system, but now a transition mechanism to tell us the probability to get from one state to another in a given time. The propagation of state for stochastic systems can frequently be presented as explicit first order (stochastic) differential equations or dilation. Here one makes idealized assumptions on the noise (for instance, that it is Wiener) and builds up the model from a mathematical basis trying to capture the real world problem as accurately as possible. The advantage has been that the models are tractable and allow for the formulation and solution of sophisticated control and optimization problems.

The programme of extending these ideas into the quantum world is an old one, with Dirac being one of the first to grasp the canonical structure behind quantum mechanics and its structural relation

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to Hamiltonian mechanics. The systematic extension to open quantum systems came about in 1970's and 80's, with the key contribution being the development of quantum stochastic calculus by Hudson and Parthasarathy,¹ see also Ref. 2, which enabled the explicit construction of dilations of completely positive quantum dynamical semi-groups.

This paper is the second in a series of papers looking at generalizations of quantum feedback networks³ where the Markov property is either pushed to its limit (depending on one's definition of Markovianity) or replaced by less idealized physical models. In the first paper of the series,⁴ we looked at models arising from quantizing transmission lines in a manner that was fundamentally non-Markov. We now turn to models which use the Hudson-Parthasarathy theory but now allow the coefficients of the quantum stochastic differential equations to be adapted quantum stochastic processes. More specifically, we want these coefficients to depend in a causal manner on a commutative quantum stochastic process which we call the control process. In this way, we obtain a controlled quantum flow when we look at the stochastic Heisenberg dynamics. What we would like to do is to perform some continuous-time measurement on the output noise from the system and take the control process to be some filtered process obtained from the measurement readout process. This is related to the theory of quantum non-demolition measurements due to Belavkin;⁵ however, our interest is with the internal consistency of the feedback mechanism and do not need explicitly the quantum filtering (trajectories) theory. In principle, the filter modulating the measurement readout process, before feeding back into the system, may be described classically and one may worry about mixing classical and quantum dynamical systems. The treatment we give however circumvents this issue.

It is not uncommon for theoretical physicists to refer to a master equation with time-dependent coupling operators as non-Markovian. As such, our model ought to be considered as radically non-Markovian as we not only allow the coupling operators to be time-dependent but also allow them to depend on the past history of the measurement readout. However, the theory we present is firmly set in the framework of the Hudson-Parthasarathy theory and makes full use of the basic methods used to build Markov models and retains all the features of being able to propagate the state forward in time, albeit with a much more complicated feedback-modulated dynamical flow. The preference of the author would be to describe these models here as pushing the quantum Markov description about as far as possible.

The use of measurement feedback in quantum optics was first proposed by Wiseman.⁶ This is an early form of what has now become classified as environment engineering. Subsequently, models were developed for realistic photo-detectors⁷ which took account of the modification of the measurement output that would likely occur in the detection—so that the real detector was effectively an ideal detector cascaded with a filter. In practice, one would want to design a filter to process the measurement readout before feeding back to the system. Here one needs the concept of a controlled dynamics, and in the context of open quantum systems, this was first formalized by Bouten and van Handel,^{8,9} and also Ref. 10. This paper builds on their framework.

The setup is depicted in Fig. 1. We have a quantum system $G \sim (S, L, H)$ with input process $B(t)$. The output field, $B^{\text{out}}(t)$, is passed to a detector which measures, say, the quadrature $Y(t) = B^{\text{out}}(t) + B^{\text{out}*}(t)$. For an excellent recent account of what is now colloquially referred to as the *SLH*-theory, see the review by Combes, Kerckhoff, and Sarovar.¹¹ The measured output is essentially classical and may be treated as a classical stochastic process—it may then be passed through a

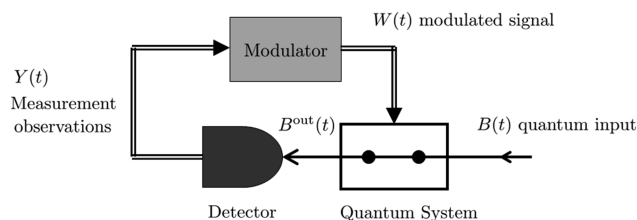


FIG. 1. The output field of a quantum system is measured and the measurement observations Y processed by a modulator. The *SLH*-coefficients may then be considered as functions of the modulated process W .

modulator which, for instance, may smooth the measured output $Y(t)$. The modulated process $W(t)$ will then be a process that depends causally on $Y(t)$ —in the case of stochastic processes, W will be adapted to the filtration generated by Y . The *SLH*-coefficients are then made to depend on $W(t)$.

The treatment here relies on making appropriately engineered interconnections between the system, the detector, and the modulator. In this respect, it is closer to the coherent quantum feedback models^{12–25} than measurement feedback;²⁶ however, we show the consistency of these approaches.

In Sec. II, we give an abstract account of controlled quantum flows where we assume that the coefficients of the quantum stochastic differential equation—the *SLH*-coefficients—are allowed to be adapted processes depending on some control process Z , which itself is an adapted commutative quantum stochastic process on the noise space. We pay particular attention to the form of the input-output relations for the noise field as this is quite different from the usual (autonomous Markov!) case where the coefficients are fixed operators on the system space. Our main contribution here is to elucidate the difference between the *input picture* and the *output picture*: in particular, the measured output, Y , should be the control process Z rotated from the input to output picture using the unitary quantum stochastic process. In Sec. III, we focus on the special case of quadrature measurement, and in this case we look at several explicit models where the measurement readout Y is modulated prior feedback to the system.

A. Quantum stochastic evolutions

The idea of stochastic dilations originates in the observation that heat equations—(hypo)-elliptic 2nd order *partial* differential equations—may be treated by instead considering 1st order *stochastic* differential equations. In a non-commutative setting, say a C^* -algebra, the role of 1st order differential operators (derivations) is played by maps of the form $[\cdot, K]$ for a fixed element K . Hudson-Parthasarathy developed a theory of quantum stochastic evolutions which gave explicit dilations of the non-commutative analogue of heat equations—Gorini, Kossakowski and Sudarshan generator (or GKS-Lindblad generator, for short)-Lindblad master equations. Their setting is a Hilbert space of the form

$$\mathfrak{H} = \mathfrak{h}_0 \otimes \Gamma(\mathfrak{K} \otimes L^2[0, \infty)), \quad (1)$$

where \mathfrak{h}_0 is a fixed Hilbert space, called the *initial space*, and \mathfrak{K} is a fixed Hilbert space called the *internal space* (or *multiplicity space*). We take a finite number, n , of boson inputs labeled by the set $\mathbf{k} = \{1, \dots, n\}$ and take $\mathfrak{K} = \mathbb{C}^{\mathbf{k}}$. The second quantization functor is denoted as $\Gamma(\cdot)$ and the theory makes extensive use of the tensor product decomposition $\Gamma(\mathfrak{k}_1 \oplus \mathfrak{k}_2) \cong \Gamma(\mathfrak{k}_1) \otimes \Gamma(\mathfrak{k}_2)$, for arbitrary Hilbert spaces \mathfrak{k}_1 and \mathfrak{k}_2 .

Taking $\{e_k : k \in \mathbf{k}\}$ to be the canonical orthonormal basis for \mathfrak{K} , then we define the k th annihilation process to be

$$B_k(t) \triangleq a(e_k \otimes 1_{[0,t]}),$$

where $a(\cdot)$ is the annihilation functor from $\mathfrak{K} \otimes L^2[0, \infty)$ to the Fock space $\Gamma(\mathfrak{K} \otimes L^2[0, \infty))$ and $1_{[0,t]}$ is the indicator function for the interval $[0, t]$. We denote its adjoint, the k th creation process, as $B_k(t)^*$. Finally, the scattering process from the k th to the j th field is defined to be

$$\Lambda_{jk}(t) \triangleq d\Gamma(|e_j\rangle\langle e_k| \otimes \pi_{[0,t]}),$$

where $d\Gamma(\cdot)$ is the differential second quantization functor and $\pi_{[0,t]}$ is the operator of pointwise multiplication by $1_{[0,t]}$ on $L^2[0, \infty)$.

The tensor product decomposition then implies the *continuous tensor product decomposition*

$$\mathfrak{H} \cong \mathfrak{H}_{[0,t]} \otimes \mathfrak{H}_{(t,\infty)}, \quad (2)$$

for each $t > 0$, where $\mathfrak{H}_{[0,t]} = \mathfrak{h}_0 \otimes \Gamma(\mathfrak{K} \otimes L^2[0, t])$ and $\mathfrak{H}_{(t,\infty)} = \Gamma(\mathfrak{K} \otimes L^2(t, \infty))$. We shall write $\mathfrak{A}_{[t]}$ for the space of bounded operators on \mathfrak{H} that act trivially on the future component $\mathfrak{H}_{(t,\infty)}$, that is,

$$\mathfrak{A}_{[t]} \equiv \mathcal{B}(\mathfrak{H}_{[0,t]}). \quad (3)$$

Following Hudson and Parthasarathy, we refer to a family $X_t = \{X_t : t \geq 0\}$ of operators on \mathfrak{H} as a *quantum stochastic process*, and we say that the process is *adapted* if $X_t \in \mathfrak{A}_{[t]}$, for each $t \geq 0$. They proceed to define quantum stochastic integrals (in the sense of Itô) of adapted processes with respect

to the creation, annihilation, scattering processes. Taking $\{x_{\alpha\beta}(t) : t \geq 0\}$ to be a family of adapted quantum stochastic processes, their quantum stochastic integral is $X_t = \int_0^t x_{\alpha\beta}(s) dB^{\alpha\beta}(s)$ which is shorthand for

$$\int_0^t x_{00}(s) ds + \sum_{j \in \mathbf{k}} \int_0^t x_{j0}(s) dB_j(s)^* + \sum_{k \in \mathbf{k}} \int_0^t x_{0k}(s) dB_k(s) + \sum_{j,k \in \mathbf{k}} \int_0^t x_{jk}(s) d\Lambda_{jk}(s).$$

The differentials are understood in the Itô sense: for each s , the coefficient is an operator in the past algebra \mathfrak{A}_{s-} while the increments are future pointing and act non-trivially on $\mathfrak{H}_{(s,\infty)}$. Given a second quantum Itô integral Y_t , with $dY_t = y_{\alpha\beta}(t) dB^{\alpha\beta}(t)$, we have the *quantum Itô product rule*

$$d(X_t \cdot Y_t) = dX_t \cdot Y_t + X_t \cdot dY_t + dX_t \cdot dY_t, \quad (4)$$

with the Itô correction given by

$$\begin{aligned} dX_t \cdot dY_t = & \sum_{k \in \mathbf{k}} \left\{ x_{0k}(t) y_{k0}(t) dt + \sum_{j \in \mathbf{k}} x_{jk}(t) y_{k0}(t) dB_j^*(t) + \sum_{l \in \mathbf{k}} x_{0k}(t) y_{kl}(t) dB_l(t) \right. \\ & \left. + \sum_{j,l \in \mathbf{k}} x_{jk}(t) y_{kl}(t) d\Lambda_{jl}^*(t) \right\} = x_{\alpha k}(t) y_{k\beta}(t) dB^{\alpha\beta}(t). \end{aligned} \quad (5)$$

The quantum Itô table gives the non-vanishing products of increments which are¹

$$\begin{aligned} d\Lambda_{ij} d\Lambda_{kl} &= \delta_{jk} d\Lambda_{il}, & d\Lambda_{ij} dB_k^* &= \delta_{jk} dB_i^*, \\ dB_i dB_k^* &= \delta_{ik} dB_l, & dB_i dB_k^* &= \delta_{ij} dt. \end{aligned}$$

For a product of n quantum stochastic integrals, we get a sum of $2^n - 1$ terms. For instance, differentials of triple products have the development

$$\begin{aligned} d(XYZ) &= (dX)YZ + X(dY)Z + XY(dZ) \\ &+ (dX)(dY)Z + (dX)Y(dZ) + X(dY)(dZ) \\ &+ (dX)(dY)(dZ). \end{aligned} \quad (6)$$

The general form of the constant operator-coefficient quantum stochastic differential equation for an adapted unitary process U is¹

$$dU(t) = \left\{ - \left(\frac{1}{2} L_k^* L_k + iH \right) dt + \sum_{j \in \mathbf{k}} L_j dB_j(t)^* - \sum_{j,k \in \mathbf{k}} L_j^* S_{jk} dB_k(t) + \sum_{j,k \in \mathbf{k}} (S_{jk} - \delta_{jk}) d\Lambda_{jk}(t) \right\} U(t), \quad (7)$$

where S_{jk} , L_j , and H are the operators on the initial Hilbert space which we collect together as

$$S_{\mathbf{k}\mathbf{k}} = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix}, \quad L_{\mathbf{k}} = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}. \quad (8)$$

Necessary and sufficient conditions for the process U to be unitary are that $S_{\mathbf{k}\mathbf{k}} = [S_{jk}]_{j,k \in \mathbf{k}}$ is unitary (i.e., $\sum_k S_{kj}^* S_{kl} = \sum_k S_{jk} S_{lk}^* = \delta_{jl} I$) and H self-adjoint. (We use the convention that $L_{\mathbf{k}} = [L_k]_{k \in \mathbf{k}}$ and that $L_{\mathbf{k}}^* L_{\mathbf{k}} = \sum_{k \in \mathbf{k}} L_k^* L_k$.) The triple (S, L, H) are termed the *Hudson-Parthasarathy parameters* of the open system evolution or more colloquially the *SLH-coefficients*.

We may write Eq. (7) as $dU(t) = dG(t) U(t)$ and we note that

$$dG(t) = - \left(\frac{1}{2} L_{\mathbf{k}}^* L_{\mathbf{k}} + iH \right) dt + \sum_{j \in \mathbf{k}} L_j dB_j(t)^* - \sum_{j,k \in \mathbf{k}} S_{jk} L_k dB_k(t) + \sum_{j,k \in \mathbf{k}} (S_{jk} - \delta_{jk}) d\Lambda_{jk}(t).$$

The isometry condition is $U^*(t) U(t) = I$ which in differential form is $(dU^*) U + U^* (dU) + (dU^*) (dU) = 0$ or equivalently

$$dG + dG^* + (dG^*) (dG) = 0.$$

Likewise the co-isometry condition is $dG + dG^* + (dG) (dG^*) = 0$.

Let X be an operator on the initial space, then we set $j_t(X) \triangleq U(t)^*[X \otimes I]U(t)$ to give its Heisenberg evolution. We refer to $\{j_t(\cdot) : t \geq 0\}$ as a *quantum stochastic flow*. From the quantum Itô calculus, we find

$$dj_t(X) = j_t(\mathcal{L}X) dt + \sum_i j_t(\mathcal{M}_i X) dB_i^*(t) + \sum_i j_t(\mathcal{N}_i X) dB_i(t) + \sum_{j,k} j_t(\mathcal{S}_{jk} X) d\Lambda_{jk}(t), \quad (9)$$

where

$$\begin{aligned} \mathcal{L}X &= \frac{1}{2} \sum_i L_i^* [X, L_i] + \frac{1}{2} \sum_i [L_i^*, X] L_i - i[X, H], \quad (\text{the Lindbladian!}), \\ \mathcal{M}_i X &= \sum_j S_{ji}^* [X, L_j], \\ \mathcal{N}_i X &= \sum_k [L_k^*, X] S_{ki}, \\ \mathcal{S}_{jk} X &= \sum_l S_{lj}^* X S_{lk} - \delta_{jk} X. \end{aligned} \quad (10)$$

Equation (9) gives the dynamical evolution of the observable $j_t(X)$ and gives the dynamics of the system driven by the external inputs. We may also include outputs by setting

$$B_k^{\text{out}}(t) \triangleq U(t)^*[I \otimes B_k(t)]U(t). \quad (11)$$

We use (6) to compute the differential of $B_k^{\text{out}}(t)$. From the quantum Itô table, we have that two of the terms vanish straight away: these are $(dU^*)(dB_j)U$ and $(dU^*)(dB_j)(dU)$, leaving

$$dB_j^{\text{out}} = U^*(dB_j)U + \left\{ (dU^*)B_j U + U^*B_j(dU) + (dU^*)B_j(dU) \right\} + U^*(dB_j)(dU).$$

The first term here is readily seen to be $U^*(dB_j)U \equiv dB_j$ since the increment is future pointing and commutes with adapted coefficients. The next three terms can be rewritten as

$$\begin{aligned} \left\{ (dU^*)B_j U + U^*B_j(dU) + (dU^*)B_j(dU) \right\} &= U^* \left\{ (dG^*)B_j + B_j(dG) + (dG^*)B_j(dG) \right\} U \\ &= U^* \left\{ (dG^*) + (dG) + (dG^*)(dG) \right\} B_j U, \end{aligned}$$

where we use the fact that $B_j(t)$ acts trivially on both the initial space and the future (beyond time t) factor of the Fock space while the increments dG and dG^* in (9) act non-trivially on these factors, so that $dG(t)$ and $dG(t)^*$ commute with $B_j(t)$ for each t . By the isometry condition (9), these three terms vanish identically. The remaining term is

$$\begin{aligned} U^*(dB_j)(dU) &= U^*(dB_j)(dG)U = U^* \left(\sum_k (S_{jk} - \delta_{jk}) dB_{jk}(t) + L_k dt \right) U \\ &= \sum_k j_t(S_{jk} - \delta_{jk}) dB_{jk}(t) + j_t(L_k) dt, \end{aligned}$$

and so we obtain

$$dB_j^{\text{out}}(t) = \sum_k j_t(S_{jk}) dB_k(t) + j_t(L_j) dt. \quad (12)$$

The output field may then be subjected to measurement. For instance, in a homodyne measurement, we may measure the quadrature process $Y_k = B_k^{\text{out}} + B_k^{\text{out}*}$. Note that in this case

$$dY_k(t) = \sum_k j_t(S_{jk}) dB_k(t) + \sum_k j_t(S_{jk}^*) dB_k^*(t) + j_t(L_j + L_j^*) dt. \quad (13)$$

B. Controlled flows

In principle, there is nothing to stop us from replacing the SLH coefficients in (7) with adapted stochastic processes $\{S_{jk}(t), L_j(t), H(t) : t \geq 0\}$. Mathematically, the conditions that $[S_{jk}(t)]_{j,k \in \mathbb{K}}$ be unitary and that $H(t)$ be self-adjoint, for all $t \geq 0$, are enough to ensure the unitarity of the corresponding process $\{U(t) : t \geq 0\}$.

To see why we might want to consider such models, let us look at the situation where S and L are fixed operators on the initial space, and where the Hamiltonian is allowed to depend on a time-varying process $Z = \{Z_t : t \geq 0\}$, say

$$H(t) \equiv h(Z_t) F, \quad (14)$$

where F is a fixed self-adjoint operator on the initial space, and $h(\cdot)$ is some real-valued function.

The corresponding unitary process may be denoted as $U_t^{[Z]}$ and may be referred to as a *controlled unitary*, specifically controlled by the process Z . By construction, the controlled unitary depends on the control, Z , in a causal manner—that is, $U_t^{[Z]}$ depends on the $\{Z_s : 0 \leq s \leq t\}$ and not on values of Z later than time t . We may take Z to be a deterministic control function; however, more generally we could take Z to be itself a quantum stochastic process, see Fig. 1. For instance, we may imagine performing a continuous measurement on the output process and feed the measured output Y back in as the control.

In Ref. 6, Wiseman considered direct feedback models where a formal Hamiltonian correction $H_{fb}(t) = F \otimes \dot{Y}(t)$ was included. Here $\dot{Y}(t)$ is the formal derivative of the measurement output process—as Y is a diffusion process, and consequently of unbounded variation, \dot{Y} needs to be interpreted with care. However, Wiseman's analysis is correct, and a rigorous way of interpreting his Hamiltonian was presented in Ref. 18 and involves a double pass through the system—first corresponding to SLH -coefficients (I, L, H) and second corresponding to $(I, -iF, 0)$. This can be viewed before feedback as the two-input two-output device with SLH -coefficients $\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} L \\ -iF \end{bmatrix}, H \right)$, that is,

$$dU(t) = \left\{ LdB_1(t)^* - L^*dB_1(t) - iF [dB_2(t) + dB_2(t)^*] - \left(\frac{1}{2}L^*L + F^2 + iH \right) dt \right\} U(t).$$

The first output is then fed in as the second input as shown in Fig. 2 to create a feedback loop. Here the term $F \otimes [dB_2(t) + dB_2(t)^*]$ is now interpreted as realizing the formal expression $H_{fb}(t) dt = F \otimes \dot{Y}(t) dt$. The closed loop system then has SLH -coefficients given by the series product

$$(I, -iF, 0) \triangleleft (I, L, H) = \left(I, L - iF, H + \frac{1}{2} (FL + L^*F) \right),$$

that is,

$$dU_{fb}(t) = \left\{ (L - iF)dB(t)^* - (L - iF)^*dB(t) - \left(\frac{1}{2}(L - iF)^*(L - iF) + iH + \frac{i}{2} (FL + L^*F) \right) dt \right\} U_{fb}(t).$$

The reduced model obtained this way fully agrees with the one derived by Wiseman.

The mathematical principle for dealing with this is quite subtle and is due to Bouten and van Handel.^{8,9} Their central observation was that one had to take care to distinguish the control process, $Z = B + B^*$, which is *fed in* as a control modifying the Hamiltonian H —more generally, S and L as

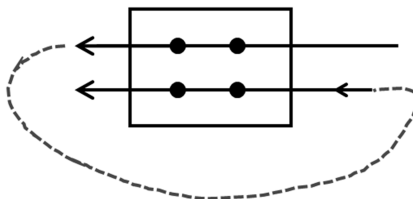


FIG. 2. A double-pass (Series Product configuration) leading to Wiseman's direct feedback.

well—and the output process $Y = B^{\text{out}} + B^{\text{out}*}$ which is the measured output that you want to somehow feed back in. The two are related by

$$Y(t) = U(t)^* Z(t) U(t). \quad (15)$$

The subtlety is that U is supposed now to be the modified $U_t^{[Z]}$ controlled by Z . In Wiseman's feedback model, the modulator takes Y and converts it into $W = \dot{Y}$ —it is described as proportional feedback, but in these terms it is arguably more a derivative controller than a proportional controller.²⁷

This brings us to the main question which we aim to resolve in this paper: when to use Z and when to use Y ? In Wiseman's derivation,⁶ the feedback Hamiltonian is $H_{\text{fb}}(t) = F \otimes \dot{Y}(t)$; however, in the Bouten and van Handel papers,^{8,9} one constructs a controlled flow with Z as the dependent process. In the derivation of Gough and James,¹⁸ there is no explicit measurement—instead there is a second pass which has the coupling operator $-iF$ with $F = F^*$ and this effectively picks out the quadrature $Z_2 = B_2 + B_2^*$. After the series product is used to make the feedback connection, everything is evolving coherently. As we shall see, there is an input picture and an output picture, and the two are unitarily equivalent. For model building, the input picture is preferable and it is here that we can apply the various interconnection rules for quantum feedback networks.³ However, the controlled quantum flow can equally well be described in the output picture, using Y , and the associated dynamical equations here are arguably more physically intuitive.

II. THE INPUT AND OUTPUT PICTURES FOR CONTROLLED OPEN DYNAMICS

Let us set out the various algebras we consider in this theory

$$\begin{aligned} \mathfrak{A}_0 &= \mathcal{B}(\mathfrak{h}_0) \text{ the initial algebra (system operators at time } t = 0) \\ \mathfrak{N} &= \mathcal{B}\left(\Gamma\left(\mathfrak{K} \otimes L^2[0, \infty)\right)\right), \text{ the noise space.} \end{aligned}$$

We note that the continuous tensor product decomposition (2) implies that $\mathfrak{N} = \mathfrak{N}_{[0,t]} \otimes \mathfrak{N}_{(t,\infty)}$, where $\mathfrak{N}_{[0,t]} = \mathcal{B}\left(\Gamma\left(\mathfrak{K} \otimes L^2[0, t]\right)\right)$, etc. The algebra of all operators up to time t , (3), is then

$$\mathfrak{A}_t = \mathfrak{A}_0 \otimes \mathfrak{N}_{[0,t]}.$$

We recall that the commutant of a set \mathfrak{A} in a von Neumann algebra \mathfrak{B} is the set $\mathfrak{A}' = \{B \in \mathfrak{B} : [A, B] = 0, \forall A \in \mathfrak{A}\}$. The von Neumann algebra generated by the set \mathfrak{A} is then the bicommutant $\text{vN}\mathfrak{A} = \mathfrak{A}'' = (\mathfrak{A}')'$.

Definition 1. A control process is a commutative family $Z = \{Z_t : t \geq 0\}$ of adapted operators acting trivially on the initial space. The filtration determined by a controlled process Z is the collection of commutative von Neumann algebras

$$\mathfrak{Z}_{[0,t]} = \text{vN}\{Z_s : 0 \leq s \leq t\} \subset \mathfrak{N}_{[0,t]},$$

for $t > 0$.

Definition 2. A controlled stochastic process $F[[Z]] = \{F_t[[Z]] : t \geq 0\}$, controlled by control process Z , is a quantum stochastic process such that $F[[Z]]$ is affiliated to $\mathfrak{Z}'_{[0,t]} \equiv \mathfrak{A}_0 \otimes \mathfrak{Z}_{[0,t]} \otimes \mathfrak{N}_{(t,\infty)}$.

We shall reserve the double square bracket notation exclusively for this situation. The operator $F_t[[Z]]$ may be thought of as a “function” taking “values” in $\mathfrak{Z}'_{[0,t]}$ with commutative “domain” $\mathfrak{Z}_{[0,t]}$. A central feature is that the “domain” space commutes with the “value” space. More correctly, we have a mapping from the spectrum of the (commutative) algebra $\mathfrak{Z}_{[0,t]}$ into its (non-commutative) commutant.

Definition 3. For a fixed control process Z , let $(S[[Z]], L[[Z]], H[[Z]])$ be controlled processes, controlled by Z , such that $S_t[[Z]]$ is unitary and $H_t[[Z]]$ is self-adjoint, for each t . The unitary process they generate, $U_t^{[Z]}$, is the solution of the quantum stochastic differential equation $dU_t^{[Z]} = dG_t^{[Z]} U_t^{[Z]}$

where

$$\begin{aligned} dG_t^{[Z]} = & - \left(\frac{1}{2} L_{k,t}[[Z]]^* L_{k,t}[[Z]] + iH_t[[Z]] \right) dt + \sum_{j \in k} L_{j,t}[[Z]] dB_j(t)^* \\ & - \sum_{j,k \in k} L_{j,t}[[Z]]^* S_{jk,t}[[Z]] dB_k(t) + \sum_{j,k \in k} (S_{jk,t}[[Z]] - \delta_{jk}) d\Lambda_{jk}(t), \end{aligned} \quad (16)$$

with $U(0) = I$.

Note that $U_t^{[Z]}$ will not generally commute with the process Z , which is why we do not use a double square-bracket notation such as $U_t^{[[Z]]}$.

In our definition of controlled flows, we required that $S_{jk,t}[[Z]]$, $L_{k,t}[[Z]]$ and $H_t[[Z]]$ are controlled processes $F_t[[Z]]$ —that is, they are affiliated to $\mathfrak{A}_0 \otimes \mathfrak{Z}_{[0,t]}$ which is a proper subset of \mathfrak{A}_t . The unitary process $U_t^{[Z]}$ they generate however is not a controlled process. It is a quantum stochastic integral whose coefficients are controlled processes but the nature of the integration will typically lead to a process in $\mathfrak{A}_0 \otimes \mathfrak{A}_{[0,t]}$, and outside $\mathfrak{A}_0 \otimes \mathfrak{Z}_{[0,t]}$ in particular. The same applies to $G_t^{[Z]}$ defined by (16) with initial condition $G_0^{[Z]} = 0$.

The identities $dG_t^{[Z]*} + dG_t^{[Z]*} + (dG_t^{[Z]}) (dG_t^{[Z]*}) = 0 = dG_t^{[Z]*} + dG_t^{[Z]*} + (dG_t^{[Z]*}) (dG_t^{[Z]})$ still hold, and the unitarity of $U_t^{[Z]}$ follows from this.

Again, let X be an operator on the initial space, then we now set $j_t^{[Z]}(X) \triangleq U_t^{[Z]*} [X \otimes I] U_t^{[Z]}$ to give its Heisenberg evolution. We refer to $\{j_t^{[Z]}(\cdot) : t \geq 0\}$ as a *controlled quantum stochastic flow*. From the quantum Itô calculus, we have

$$\begin{aligned} dj_t^{[Z]}(X) = & U_t^{[Z]*} \left(\mathcal{L}_t^{[[Z]]} X \right) U_t^{[Z]} dt + \sum_i U_t^{[Z]*} \left(\mathcal{M}_{i,t}^{[[Z]]} X \right) U_t^{[Z]} dB_i(t)^* \\ & + \sum_i U_t^{[Z]*} \left(\mathcal{N}_{i,t}^{[[Z]]} X \right) U_t^{[Z]} dB_i(t) + \sum_{j,k} U_t^{[Z]*} \left(\mathcal{S}_{jk,t}^{[[Z]]} X \right) U_t^{[Z]} d\Lambda_{jk}(t), \end{aligned} \quad (17)$$

where the new super-operators $\mathcal{L}_t^{[[Z]]}$, $\mathcal{M}_{i,t}^{[[Z]]}$, $\mathcal{N}_{i,t}^{[[Z]]}$, $\mathcal{S}_{jk,t}^{[[Z]]}$ are just the same for as in (10) with the *SLH*-coefficients now replaced by the controlled versions $S_t[[Z]]$, $L_t[[Z]]$, $H_t[[Z]]$.

Likewise, the output processes are now $B_k^{\text{out},[Z]}(t) \triangleq U_t^{[Z]*} [I \otimes B_k(t)] U_t^{[Z]}$. We may again use the quantum Itô calculus as before to derive the analogue of (12). We find that the first term should be $U_t^{[Z]*} dB_j(t) U_t^{[Z]}$ which again equals $dB_j(t)$. However, the argument we previously used to show that the next group of three terms vanishes breaks down since $B_j(t)$ no longer commutes with $dG_t^{[Z]}$ due to its possible dependence on Z_s for $0 \leq s \leq t$.

Instead we obtain

$$\begin{aligned} dB_j^{\text{out}}(t) = & \sum_k U_t^{[Z]*} (S_{jk,t}[[Z]]) U_t^{[Z]} dB_k(t) + U_t^{[Z]*} (L_{j,t}[[Z]]) U_t^{[Z]} dt \\ & + U_t^{[Z]*} \left\{ (dG_t^{[Z]*}) B_j(t) + B_j(t) (dG_t^{[Z]}) + (dG_t^{[Z]*}) B_j(t) (dG_t^{[Z]}) \right\} U_t^{[Z]}. \end{aligned} \quad (18)$$

The term in braces in (18) may be written as

$$\begin{aligned} & \left\{ \frac{1}{2} \sum_i L_{i,t}[[Z]]^* [B_j(t), L_{i,t}[[Z]]] + \frac{1}{2} \sum_i [L_{i,t}[[Z]]^*, B_j(t)] L_{i,t}[[Z]] - i [B_j(t), H_t[[Z]]] \right\} dt \\ & + \sum_{i,k} S_{ki,t}[[Z]]^* [B_j(t), L_{k,t}[[Z]]] dB_i(t)^* \\ & + \sum_{i,k} [L_{k,t}[[Z]]^*, B_j(t)] S_{ki,t}[[Z]] dB_i(t) \\ & + \sum_{i,k} \left(\sum_l S_{li,t}[[Z]]^* B_j(t) S_{lk,t}[[Z]] - \delta_{ik} B_j(t) \right) d\Lambda_{ik}(t). \end{aligned} \quad (19)$$

Previously this vanishes since the $B_j(t)$ fields lived in the noise algebra while the *SLH* coefficients lived in the initial algebra—this time the *SLH* coefficients are themselves adapted processes.

A. The control and the measurement algebra

Given an adapted unitary process $U(\cdot)$, we obtain the measurement algebras

$$\mathfrak{Y}_{[0,t]} = U(t)^* \mathfrak{Z}_{[0,t]} U(t) = \vee N \{Y_s : 0 \leq s \leq t\},$$

where $Y_t = U(t)^* Z_t U(t)$. (Note that $Y_s \equiv U(t)^* Z_s U(t)$ whenever $t \geq s$.) The algebras $\mathfrak{Y}_{[0,t]}$ are again commutative and form a filtration of von Neumann algebras.

Observation 4. Let $U(\cdot)$ be an adapted unitary process and define

$$\tilde{F}_t[[Y]] = U(t)^* F_t[[Z]] U(t).$$

Let us set $\mathfrak{A}_t = U(t)^* \mathfrak{A}_0 U(t)$ which is the initial algebra unitarily rotated by $U(t)$, and then $\mathfrak{N}_{[0,t]}^{\text{out}} = U(t)^* \mathfrak{N}_{[0,t]} U(t)$ which is the noise algebra unitarily rotated by $U(t)$ the total output algebra. Note that the measurement algebra $\mathfrak{Y}_{[0,t]}$ is a (commutative) subalgebra of $\mathfrak{N}_{[0,t]}^{\text{out}}$. The rotated process $\tilde{F}_t[[Y]]$ takes values in $U(t)^* \mathfrak{A}_t U(t) \cong \mathfrak{A}_t \otimes \mathfrak{Y}_{[0,t]}$ and has domain $\mathfrak{Y}_{[0,t]}$, and again we have the central feature is that the domain space commutes with the value space. This, in fact, is our justification for writing it in the form $\tilde{F}_t[[Y]]$.

Notation 5. We now take the unitary process U performing the rotation to be the controlled unitary $U^{[Z]}$. In the following, we will assume this choice and set

$$\tilde{F}_t[[Y]] \triangleq U_t^{[Z]*} F_t[[Z]] U_t^{[Z]}. \quad (20)$$

For the special case where $F_t[[Z]] \equiv X \otimes I$, with $X \in \mathfrak{A}_0$, we will also write

$$j_t^{[[Y]]}(X) \triangleq U_t^{[Z]*} [X \otimes I] U_t^{[Z]}, \quad (21)$$

for $F_t[[Z]] \equiv I \otimes B(t)$, we write

$$B_k^{[[Y]]\text{out}}(t) \triangleq U_t^{[Z]*} [I \otimes B_k(t)] U_t^{[Z]}, \quad (22)$$

and similarly $\Lambda_{jk}^{[[Y]]\text{out}}(t) \triangleq U_t^{[Z]*} [I \otimes \Lambda_{jk}(t)] U_t^{[Z]}$. We refer to the description in terms of the control process Z as the input picture and the description in terms of the measurement readout process Y as the output picture.

Note that (21) and (22) were previously denoted as $j_t^{[Z]}(X)$ and $B_k^{[Y]\text{out}}(t)$, respectively. The reader may well have noticed that sometimes we use single square brackets and sometimes double—to recap, we use single brackets to indicate a dependence of an operator-valued process on a commutative control process but use double square brackets to indicate an operator-valued function with a commutative domain whenever the domain and range variables commute.

Proposition 6. (Non-demolition Principle for Controlled Flows) For each $s \leq t$ and $X \in \mathfrak{A}_0$, we have that Y_s will commute with $j_t^{[[Y]]}(X)$.

Proof. We define the two-parameter family $\{U_{t,s}^{[Z]} : t \geq s\}$ by $U_{t,s}^{[Z]} = I + \int_s^t dG_\tau^{[Z]} U_{\tau,s}^{[Z]}$. We see that $U_t^{[Z]} \equiv U_{t,0}^{[Z]}$, and that $U_t^{[Z]} \equiv U_{t,s}^{[Z]} U_s^{[Z]}$. In particular, Z_s commutes with $U_{t,s}^{[Z]}$ for $t \geq s$, since $dG_\tau[[Z]]$ commutes with Z_s for all $s \leq \tau \leq t$: this implies that for $t \geq s$,

$$U_t^{[Z]*} [I \otimes Z_s] U_t^{[Z]} = U_s^{[Z]*} U_{t,s}^{[Z]*} [I \otimes Z_s] U_{t,s}^{[Z]} U_s^{[Z]} = Y_s.$$

Therefore, for $t \geq s$,

$$[j_t^{[[Y]]}(X), Y_s] = U_t^{[Z]*} [X \otimes I, I \otimes Z_s] U_s^{[Z]} = 0. \quad \blacksquare$$

While the proof is similar to that for standard uncontrolled flows, see Ref. 10, however we note that in the latter situation we also have the separate identities $[j_t(X), B_k^{\text{out}}(t)] = [j_t(X), B_k^{\text{out}*}(t)] = [j_t(X), \Lambda_{jk}^{\text{out}}(t)] = 0$, which need not necessarily hold true for controlled flows.

Lemma 7. Let us fix a control process Z and take $U^{[Z]}$ to be the unitary evolution generated by the controlled SLH coefficient processes $(S[[Z]], L[[Z]], H[[Z]])$. Then $U_t^{[Z]} \equiv V_t^{[Y]}$ where $V_t^{[Y]}$ is the solution to the quantum stochastic differential equation

$$dV_t^{[Y]} = V_t^{[Y]} d\tilde{G}_t^{[Y]}, \quad V_0^{[Y]} = I, \quad (23)$$

where

$$\begin{aligned} d\tilde{G}_t^{[Y]} = & - \left(\frac{1}{2} \tilde{L}_{k,t}[[Y]]^* \tilde{L}_{k,t}[[Y]] + i\tilde{H}_t[[Y]] \right) dt + \sum_{j \in \mathbf{k}} \tilde{L}_{j,t}[[Y]] dB_j(t)^* \\ & - \sum_{j,k \in \mathbf{k}} \tilde{S}_{jk,t}[[Y]] \tilde{L}_{k,t}[[Y]] dB_k(t) + \sum_{j,k \in \mathbf{k}} (\tilde{S}_{jk,t}[[Y]] - \delta_{jk}) d\Lambda_{jk}(t). \end{aligned} \quad (24)$$

Proof. Whilst this is one of the key observations we need from a conceptual point of view, its proof is actually quite trivial. The quantum stochastic differential equation for the unitary $U^{[Z]}$ is easily rearranged to read as

$$dU_t^{[Z]} = dG_t^{[Z]} U_t^{[Z]} = U_t^{[Z]} U_t^{[Z]*} dG_t^{[Z]} U_t^{[Z]} = U_t^{[Z]} d\tilde{G}_t^{[Y]},$$

which is the same one as $V_t^{[Y]}$ with the same initial condition. ■

B. The controlled stochastic Heisenberg dynamics

For a fixed operator X in the initial algebra, we have introduced the evolution $j_t^{[Z]}(X) = U_t^{[Z]*} (X \otimes I) U_t^{[Z]}$. This is clearly a special case of (20) with $F_t[[Z]] = X \otimes I$. We could therefore write $j_t^{[Z]}(X)$ as $X_t[[Z]]$ or equally well as $\tilde{X}_t[[Y]]$. The Heisenberg-Langevin Eq. (17) then becomes

$$\begin{aligned} d\tilde{X}_t[[Y]] = & \left\{ \frac{1}{2} \sum_i \tilde{L}_{i,t}[[Y]]^* \left[\tilde{X}_t[[Y]], \tilde{L}_{i,t}[[Y]] \right] + \frac{1}{2} \sum_i \left[\tilde{L}_{i,t}[[Y]]^*, \tilde{X}_t[[Y]] \right] \tilde{L}_{i,t}[[Y]] \right. \\ & \left. - i \left[\tilde{X}_t[[Y]], \tilde{H}_t[[Y]] \right] \right\} dt \\ & + \sum_{i,k} \tilde{S}_{ki,t}[[Y]]^* \left[\tilde{X}_t[[Y]], \tilde{L}_{k,t}[[Y]] \right] dB_i^*(t) \\ & + \sum_{i,k} \left[\tilde{L}_{k,t}[[Y]]^*, \tilde{X}_t[[Y]] \right] \tilde{S}_{ki,t}[[Y]] dB_i(t) \\ & + \sum_{i,k} \left(\sum_l \tilde{S}_{li,t}[[Y]]^* \tilde{X}_t[[Y]](t) \tilde{S}_{lk,t}[[Y]] - \delta_{ik} \tilde{X}_t[[Y]] \right) d\Lambda_{ik}(t). \end{aligned} \quad (25)$$

C. The controlled system input-output relations

The output is given as $B_j^{\text{out}[Z]}(t) = U_t^{[Z]*} (I \otimes B_j(t)) U_t^{[Z]}$. Let us write this as $\tilde{B}_j^{\text{out}[Y]}(t)$, then from (18) and (19) we have

$$\begin{aligned} d\tilde{B}_j^{\text{out}[Y]}(t) = & \sum_k \tilde{S}_{jk,t}[[Y]] dB_j(t) + \tilde{L}_{j,t}[[Y]] dt \\ & + \left\{ \frac{1}{2} \sum_i \tilde{L}_{i,t}[[Y]]^* \left[\tilde{B}_j^{\text{out}[Y]}(t), \tilde{L}_{i,t}[[Y]] \right] + \frac{1}{2} \sum_i \left[\tilde{L}_{i,t}[[Y]]^*, \tilde{B}_j^{\text{out}[Y]}(t) \right] \tilde{L}_{i,t}[[Y]] \right. \\ & \left. - i \left[\tilde{B}_j^{\text{out}[Y]}(t), \tilde{H}_t[[Y]] \right] \right\} dt \\ & + \sum_{i,k} \tilde{S}_{ki,t}[[Y]]^* \left[\tilde{B}_j^{\text{out}[Y]}(t), \tilde{L}_{k,t}[[Y]] \right] dB_i^*(t) \\ & + \sum_{i,k} \left[\tilde{L}_{k,t}[[Y]]^*, \tilde{B}_j^{\text{out}[Y]}(t) \right] \tilde{S}_{ki,t}[[Y]] dB_i(t) \\ & + \sum_{i,k} \left(\sum_l \tilde{S}_{li,t}[[Y]]^* \tilde{B}_j^{\text{out}[Y]}(t) \tilde{S}_{lk,t}[[Y]] - \delta_{ik} \tilde{B}_j^{\text{out}[Y]}(t) \right) d\Lambda_{ik}(t). \end{aligned} \quad (26)$$

III. QUADRATURE FEEDBACK

Let us consider a homodyne measurement scheme where we measure the quadrature process. Here we set

$$Z(t) = B(t) + B(t)^*$$

(for simplicity we consider only a single input single output model, $n = 1$). The measured output is then

$$Y(t) = U_t^{[Z]*} [B(t) + B(t)^*] U_t^{[Z]} = \tilde{B}^{\text{out}}(t) + \tilde{B}^{\text{out}}(t)^*. \quad (27)$$

In the present case ($n = 1$), Eq. (26) reduces to

$$\begin{aligned} d\tilde{B}^{\text{out}}(t) = & \tilde{S}_t dB(t) + \tilde{L}_t dt \\ & + \left\{ \frac{1}{2} \tilde{L}_t^* [\tilde{B}^{\text{out}}(t), \tilde{L}_t] + \frac{1}{2} [\tilde{L}_t^*, \tilde{B}^{\text{out}}(t)] \tilde{L}_t - i [\tilde{B}^{\text{out}}(t), \tilde{H}_t] \right\} dt \\ & + \tilde{S}_t^* [\tilde{B}^{\text{out}}(t), \tilde{L}_t] dB(t)^* + [\tilde{L}_t^*, \tilde{B}^{\text{out}}(t)] \tilde{S}_t dB(t) + \left(\tilde{S}_t^* \tilde{B}^{\text{out}}(t) \tilde{S}_t - \tilde{B}^{\text{out}}(t) \right) d\Lambda(t), \end{aligned} \quad (28)$$

where we drop the Y -dependence for convenience.

When dealing with terms such as $[\tilde{B}^{\text{out}[Y]}(t), \tilde{H}_t[[Y]]]$, we would like to know whether we remain within the algebra $\mathfrak{Y}_{[0,t]}$. We note that this may be written as $U_t^{[Z]} [I \otimes B(t), H_t[[Z]]] U_t^{[Z]*}$, so it is enough to show that $[I \otimes B(t), H_t[[Z]]]$ remains in $\mathfrak{A}_0 \otimes \mathfrak{Z}_{[0,t]}$. The following Lemma gives an affirmative answer.

Lemma 8. Let $F_t[[Z]]$ be a stochastic process controlled by the quadrature process Z . Then the commutator $[I \otimes B(t), F_t[[Z]]]$ is affiliated to $\mathfrak{A}_0 \otimes \mathfrak{Z}_{[0,t]}$.

Proof. We may assume that the process $F_t[[Z]]$ admits a chaotic expansion of the form

$$F_t[[Z]] = \sum_{n=0}^{\infty} \int_{\Delta_n(t)} F(\tau_n, \dots, \tau_1) \otimes dZ_{\tau_n} \dots dZ_{\tau_1},$$

where $\Delta_n(t)$ is the simplex $\{(\tau_n, \dots, \tau_1) : t \geq \tau_n > \dots > \tau_1 \geq 0\}$. The commutation relations for the creation and annihilation processes are $[B_t, B_s^*] = t \wedge s I$, where $t \wedge s$ is the minimum of t and s . Under the integral sign, we see that

$$[B_t, dZ_{\tau}] \equiv d\tau, \quad (\tau < t),$$

and therefore

$$[I \otimes B(t), F_t[[Z]]] = \sum_{n=1}^{\infty} \sum_{k=1}^n \int_{\Delta_n(t)} F(\tau_n, \dots, \tau_1) \otimes dZ_{\tau_n} \dots d\tau_k \dots dZ_{\tau_1}.$$

We see that the commutator is evidently affiliated with $\mathfrak{A}_0 \otimes \mathfrak{Z}_{[0,t]}$. ■

Proposition 9. Equation (27) implies that

$$dY(t) = \tilde{S}_t dB(t) + \tilde{S}_t^* dB(t)^* + (\tilde{L}_t + \tilde{L}_t^*) dt. \quad (29)$$

Proof. We now substitute (28) into $dY(t) = d\tilde{B}^{\text{out}}(t) + d\tilde{B}^{\text{out}}(t)^*$ —we find that the first two terms combine to give (29) while the other terms vanish. To see why, let us examine the coefficient of the $d\Lambda$ term

$$\left(\tilde{S}_t^* \tilde{B}^{\text{out}}(t) \tilde{S}_t - \tilde{B}^{\text{out}}(t) \right) + \left(\tilde{S}_t^* \tilde{B}^{\text{out}}(t) \tilde{S}_t - \tilde{B}^{\text{out}}(t) \right)^* \equiv \left(\tilde{S}_t^* Y(t) \tilde{S}_t - Y(t) \right),$$

but $\tilde{S}_t = \tilde{S}_t[[Y]]$ commutes with $Y(t)$ by construction. Similarly, with the combination of $d\tilde{B}^{\text{out}}(t)$ and $d\tilde{B}^{\text{out}}(t)^*$, the coefficients reconstitute as commutators of $Y(t) = \tilde{B}^{\text{out}}(t) + \tilde{B}^{\text{out}}(t)^*$ with $\tilde{L}_t[[Y]]$ and $\tilde{H}_t[[Y]]$, and these likewise vanish identically. ■

A. Feedback to the Hamiltonian

We now look at several examples.

1. Proportional feedback

Here we just take $(W_t = Z_t)$ and choose the SLH coefficients to be

$$S_t[[Z]] = S, \quad L_t[[Z]] = L, \quad H_t[[Z]] = F \otimes Z_t,$$

where S , L , and $F = F^*$ are fixed operators on the system space \mathfrak{h}_0 . We find that

$$dB^{\text{out}}(t) = j_t^{[[Y]]}(S) dB(t) + j_t^{[[Y]]}(L - itF) dt,$$

which follows from (26) and the observation that $[I \otimes B(t), H_t[[Z]]] = F \otimes tI$. From (25), the Heisenberg equations are then

$$\begin{aligned} dj_t^{[[Y]]}(X) = & \left\{ j_t^{[[Y]]}(\mathcal{L}(X)) - i j_t^{[[Y]]}([X, F]) Y_t \right\} dt \\ & + j_t^{[[Y]]}(\mathcal{M}(X)) dB^*(t) + j_t^{[[Y]]}(\mathcal{N}(X)) dB(t) + j_t^{[[Y]]}(\mathcal{S}(X)) d\Lambda^*(t). \end{aligned}$$

2. Nonlinear modulator

We may take the modulated process to be $(W_t = g(Z_t))$, so that

$$S_t[[Z]] = S, \quad L_t[[Z]] = L, \quad H_t[[Z]] = F \otimes g(Z_t),$$

where now g is some nonlinear function. This time we have $[I \otimes B(t), H_t[[Z]]] = F \otimes t g'(Z_t)$ and so

$$dB^{\text{out}}(t) = j_t^{[[Y]]}(S) dB(t) + \left(j_t^{[[Y]]}(L) - it j_t^{[[Y]]}(F) g'(Y_t) \right) dt.$$

(Note that $g'(Y_t)$ will commute with $j_t^{[[Y]]}(F)$, etc.) From (25), the Heisenberg equations become

$$\begin{aligned} dj_t^{[[Y]]}(X) = & \left\{ j_t^{[[Y]]}(\mathcal{L}(X)) - i j_t^{[[Y]]}([X, F]) g(Y_t) \right\} dt \\ & + j_t^{[[Y]]}(\mathcal{M}(X)) dB^*(t) + j_t^{[[Y]]}(\mathcal{N}(X)) dB(t) + j_t^{[[Y]]}(\mathcal{S}(X)) d\Lambda^*(t). \end{aligned}$$

3. Causal linear filter modulator

More generally, the modulator may act as a causal linear filter, say

$$S_t[[Z]] = S, \quad L_t[[Z]] = L, \quad H_t[[Z]] = \int_0^t F(t-s) \otimes dZ_s,$$

where $F(\cdot)$ is a self-adjoint \mathfrak{A}_0 -valued function of time. (For the special case $F(t) = F h(t)$, with $F = F^* \in \mathfrak{A}_0$ and $h(\cdot)$ a fixed real-valued function, we obtain $H_t[[Z]] = F \otimes W_t$ where $W_t = \int_0^t h(t-s) \otimes dZ_s$ is a convolution.) This time we have $[I \otimes B(t), H_t[[Z]]] = \int_0^t F(t-s) \otimes ds \equiv \int_0^t F(u) \otimes du$, and so

$$dB^{\text{out}}(t) = j_t^{[[Y]]}(S) dB(t) + \left(j_t^{[[Y]]}(L) - i \int_0^t j_t^{[[Y]]}(F_u) du \right) dt.$$

From (25), the Heisenberg equations are then

$$\begin{aligned} dj_t^{[[Y]]}(X) = & \left\{ j_t^{[[Y]]}(\mathcal{L}(X)) - i \int_0^t j_t^{[[Y]]}([X, F_{t-s}]) dY_s \right\} dt \\ & + j_t^{[[Y]]}(\mathcal{M}(X)) dB^*(t) + j_t^{[[Y]]}(\mathcal{N}(X)) dB(t) + j_t^{[[Y]]}(\mathcal{S}(X)) d\Lambda^*(t). \end{aligned}$$

(Again, note that the integrand $j_t^{[[Y]]}([X, F_{t-s}])$ commutes with the increment dY_s .)

B. Feedback to the coupling operator

Let us consider a cavity mode a with the SLH -coefficients

$$S_t[[Z]] = e^{i\theta}, \quad L_t[[Z]] = \sqrt{\gamma}a \otimes I + \lambda I \otimes Z_t, \quad H = \omega a^* a.$$

We denote the time-evolved mode as $\tilde{a}_t = j_t^{[Y]}(a) = U_t^{[Z]*}(a \otimes I)U_t^{[Z]}$, then the Heisenberg equations and input-output equations read as

$$\begin{aligned} d\tilde{a}_t &= -\frac{1}{2}(\sqrt{\gamma}\tilde{a}_t + \lambda Y_t)dt - i\omega\tilde{a}_t - \sqrt{\gamma}dB(t), \\ dY_t &= e^{i\theta}dB(t) + e^{-i\theta}dB(t)^* + [\sqrt{\gamma}(\tilde{a}_t + \tilde{a}_t^*) + 2\lambda Y_t]dt, \end{aligned}$$

which may be written as the *linear* differential equation $dx_t = Ax_t dt + Bdu_t$ where

$$x_t = \begin{bmatrix} \tilde{a}_t \\ \tilde{a}_t^* \\ Y_t \end{bmatrix}, \quad du_t = \begin{bmatrix} dB(t) \\ dB(t)^* \end{bmatrix}, \quad A = \begin{bmatrix} -(\frac{1}{2}\gamma + i\omega) & 0 & -\frac{1}{2}\sqrt{\gamma}\lambda \\ 0 & -(\frac{1}{2}\gamma - i\omega) & -\frac{1}{2}\sqrt{\gamma}\lambda \\ \sqrt{\gamma} & \sqrt{\gamma} & 2\lambda \end{bmatrix}, \quad B = \begin{bmatrix} -\sqrt{\gamma}0 & 0 \\ 0 & -\sqrt{\gamma} \\ e^{i\theta} & e^{-i\theta} \end{bmatrix}.$$

The solution is then $x_t = e^{At}x_0 + \int_0^t e^{A(t-s)}Bdu_s$. The matrix A has determinant $2\omega^2\lambda$ which is non-degenerate for $\lambda \neq 0$ provided $\omega \neq 0$.

We will concentrate on the case $\omega = 0$, where the eigenvalues of A are readily calculated to be $0, -\frac{1}{2}\gamma, -\frac{1}{2}\gamma + 2\lambda$. We see that A is (marginally) stable provided that $\lambda < \frac{1}{4}\gamma$. The solution is

$$\tilde{a}_t = f(t)a + g(t)a^* + \int_0^t (e^{i\theta}k(t-s) - \sqrt{\gamma}f(t-s))dB(s) + \int_0^t (e^{-i\theta}k(t-s) - \sqrt{\gamma}g(t-s))dB(s)^*,$$

with

$$\begin{aligned} f(t) &= -\frac{2\lambda}{\gamma - 4\lambda} + \frac{\gamma}{\gamma - 2\lambda}e^{-(\frac{1}{2}\gamma - 2\lambda)t} + \frac{1}{2}e^{-\frac{1}{2}\gamma t}, \\ g(t) &= -\frac{2\lambda}{\gamma - 4\lambda} + \frac{\gamma}{\gamma - 2\lambda}e^{-(\frac{1}{2}\gamma - 2\lambda)t} - \frac{1}{2}e^{-\frac{1}{2}\gamma t}, \\ k(t) &= \frac{\sqrt{\gamma}\lambda}{\gamma - 4\lambda} \left(e^{-(\frac{1}{2}\gamma - 2\lambda)t} - 1 \right). \end{aligned}$$

We also find that the measurement process can be given the explicit form

$$Y_t = \int_0^t (e^{i\theta}r(t-s) + \gamma p(t-s))dB(s) + \int_0^t (e^{-i\theta}r(t-s) + \gamma p(t-s))dB(s)^*,$$

with

$$p(t) = -\frac{2}{\gamma - 4\lambda} \left(1 - e^{-(\frac{1}{2}\gamma - 2\lambda)t} \right), \quad r(t) = -\frac{1}{\gamma - 4\lambda} \left(\gamma - 4\lambda e^{-(\frac{1}{2}\gamma - 2\lambda)t} \right).$$

IV. PHOTON NUMBER FEEDBACK

An alternative option is to measure the photon number of the output field, $\Lambda^{\text{out}}(t)$. (For convenience, we will treat the $n = 1$ input field case.) This means that we set $Z_t \equiv \Lambda(t)$. In the case where the flow is determined by fixed SLH -components on the initial algebra \mathfrak{A}_0 , we have

$$d\Lambda^{\text{out}}(t) = d\Lambda(t) + j_t(S^*L)dB^*(t) + j_t(L^*S)dB(t) + j_t(L^*L)dt. \quad (30)$$

We now show that we obtain the same sort of consistency result we had for quadrature measurements from Proposition 9.

Proposition 10. Using the same notation as in Proposition 9 and taking $Z \equiv \Lambda$, we have that the output number operator for the controlled dynamics is

$$d\Lambda^{\text{out}[Y]}(t) \equiv d\Lambda(t) + \tilde{S}_t^* \tilde{L}_t dB^*(t) + \tilde{L}_t^* \tilde{S}_t dB(t) + \tilde{L}_t^* \tilde{L}_t dt. \quad (31)$$

Proof. A simple application of the quantum Itô calculus shows that, for a controlled flow, the analogue of (26) is

$$\begin{aligned}
 d\Lambda^{\text{out}[Y]}(t) = & d\Lambda(t) + \tilde{S}_t^* \tilde{L}_t dB^*(t) + \tilde{L}_t^* \tilde{S}_t dB(t) + \tilde{L}_t^* \tilde{L}_t dt \\
 & + \left\{ \frac{1}{2} \tilde{L}_t [[Y]]^* \left[\tilde{\Lambda}^{\text{out}[Y]}(t), \tilde{L}_t [[Y]] \right] + \frac{1}{2} \sum_i \left[\tilde{L}_t [[Y]]^*, \tilde{\Lambda}^{\text{out}[Y]}(t) \right] \tilde{L}_t [[Y]] \right. \\
 & \left. - i \left[\tilde{\Lambda}^{\text{out}[Y]}(t), \tilde{H}_t [[Y]] \right] \right\} dt \\
 & + \tilde{S}_t [[Y]]^* \left[\tilde{\Lambda}^{\text{out}[Y]}(t), \tilde{L}_t [[Y]] \right] dB^*(t) \\
 & + \left[\tilde{L}_t [[Y]]^*, \tilde{\Lambda}^{\text{out}[Y]}(t) \right] \tilde{S}_t [[Y]] dB(t) \\
 & + \left(\tilde{S}_t [[Y]]^* \tilde{\Lambda}^{\text{out}[Y]}(t) \tilde{S}_t [[Y]] - \tilde{\Lambda}^{\text{out}[Y]}(t) \right) d\Lambda(t).
 \end{aligned}$$

Fortunately, the new terms vanish for a fairly simple reason. If we take one of the terms, say $\left[\tilde{L}_t [[Y]]^*, \tilde{\Lambda}^{\text{out}[Y]}(t) \right]$, then we note that this corresponds to

$$\left[\tilde{L}_t [[Y]]^*, \tilde{\Lambda}^{\text{out}[Y]}(t) \right] = U_t^{[Z]*} \left[L_t [[Z]]^*, \Lambda(t) \right] U_t^{[Z]},$$

but the almost trivial observation at this point is that now $Z \equiv \Lambda$, and so $\Lambda(t)$ will commute with $S_t [[Z]]$, $L_t [[Z]]$, $H_t [[Z]]$ and their adjoints. This leaves us with (31) as claimed. ■

We see that (31) is structurally identical to (30).

The Heisenberg equations under the controlled flow however will be identical to those derived for the quadrature case, but with Y now an inhomogeneous Poisson process rather than a diffusion.

V. QUANTUM PID FILTER

In this section, we show how to describe one of the basic control feedback loop mechanisms, proportional-integral-differential (PID) controllers, in the quantum domain. In its classical form, we have a modulated control signal of the form

$$\dot{W}_t = k_P Y_t + k_I \int_0^t Y_s ds + k_D \dot{Y}_t,$$

which is the sum of three terms: one *proportional* to Y , one an *integral* of Y , and one the *derivative* of Y . The proportional and integral terms can be modeled following the theory set out in this paper. The derivative term however is more singular and has to be treated separately.

A. Quadrature feedback

We consider the case $Y_t = B_t^{\text{out}} + B_t^{\text{out}*}$ corresponding to quadrature measurement, and so we set $Z_t = B_t + B_t^*$ in the input picture. We shall replace the coefficients k_P, k_I, k_D with self-adjoint operators F_P, F_I , and F_D , respectively, in \mathfrak{A}_0 .

Our choice for the adapted *SLH* coefficients will be

$$\begin{aligned}
 S_t [[Z]] &= I \otimes I, \\
 L_t [[Z]] &= L_0 \otimes I - iF_D \otimes I, \\
 H_t [[Z]] &= H_0 \otimes I + \frac{1}{2} (F_D L_0 + L_0^* F_D) \otimes I + F_P \otimes Z_t + F_I \otimes \int_0^t Z_s ds.
 \end{aligned}$$

The basis for this is that the derivative term is treated in the same way as in our description of Wiseman's proportional to \dot{Y} feedback: it results in an additional term in the L -operator and an addition to the Hamiltonian. We may view this as a bare model $G_0 \sim (I, L_0, H_0)$ into which we feedback the measurement process.

For the output noise, we have $dB^{\text{out}}(t) = dB(t) + j_t^{[[Y]]}(L_0 - iF_D)dt$ so that $dY_t = dZ_t + j_t^{[[Y]]}(L_0 + L_0^*)dt$: that is, the F_D terms vanish in accordance with our consistency results from earlier.

The Heisenberg dynamics is then

$$dj_t^{[[Y]]}(X) = j_t^{[[Y]]}(\mathcal{L}X)dt - ij_t^{[[Y]]}([X, F_P])Y_tdt - ij_t^{[[Y]]}([X, F_I])\left(\int_0^t Y_s ds\right)dt \\ + j_t^{[[Y]]}([X, L_0 - iF_D])dB_t^* + j_t^{[[Y]]}([L_0^* + iF_D, X])dB_t,$$

where

$$\mathcal{L}X = \frac{1}{2}(L_0^* + iF_D)[X, L_0 - iF_D] + \frac{1}{2}[L_0^* + iF_D, X](L_0 - iF_D) - i\left[X, H_0 + \frac{1}{2}F_DL_0 + \frac{1}{2}L_0^*F_D\right] \\ \equiv \mathcal{L}_0X - \frac{1}{2}[[X, F_D], F_D] + i[F_D, X]L_0 - iL_0^*[X, F_D].$$

Here \mathcal{L}_0 is the bare GKS-Lindblad generator determined by coupling L_0 and Hamiltonian H_0 . With this we may write

$$dj_t^{[[Y]]}(X) = j_t^{[[Y]]}(\mathcal{L}_0X)dt - \frac{1}{2}j_t^{[[Y]]}([X, F_D], F_D)dt - ij_t^{[[Y]]}([X, F_P])Y_tdt \\ - ij_t^{[[Y]]}([X, F_I])\left(\int_0^t Y_s ds\right)dt \\ - i(dB_t + j_t^{[[Y]]}(L_0)dt)^* j_t^{[[Y]]}([X, F_D]) - ij_t^{[[Y]]}([X, F_D])\left(dB_t + j_t^{[[Y]]}(L_0)dt\right) \\ + j_t^{[[Y]]}([X, L_0])dB_t^* + j_t^{[[Y]]}([L_0^*, X])dB_t.$$

Note that

$$-i(dB_t + j_t^{[[Y]]}(L_0)dt)^* j_t^{[[Y]]}([X, F_D]) - ij_t^{[[Y]]}([X, F_D])\left(dB_t + j_t^{[[Y]]}(L_0)dt\right) \\ = -i(dB_t^{\text{out}} + j_t^{[[Y]]}(F_D)dt)^* j_t^{[[Y]]}([X, F_D]) - ij_t^{[[Y]]}([X, F_D])\left(dB_t^{\text{out}} + j_t^{[[Y]]}(F_D)dt\right) \\ = -ij_t^{[[Y]]}([X, F_D])\left(dB_t + dB_t^{\text{out}*}\right) + j_t^{[[Y]]}([X, F_D], F_D)dt \\ = -ij_t^{[[Y]]}([X, F_D])dY_t + j_t^{[[Y]]}([X, F_D], F_D)dt,$$

so that we obtain

$$dj_t^{[[Y]]}(X) = j_t^{[[Y]]}(\mathcal{L}_0X)dt + j_t^{[[Y]]}([X, L_0])dB_t^* + j_t^{[[Y]]}([L_0^*, X])dB_t \\ + \frac{1}{2}j_t^{[[Y]]}([X, F_D], F_D)dt - i\left\{j_t^{[[Y]]}([X, F_P])Y_tdt + j_t^{[[Y]]}([X, F_I])\left(\int_0^t Y_s ds\right)dt \right. \\ \left. + j_t^{[[Y]]}([X, F_D])dY_t\right\}. \quad (32)$$

We therefore obtain the bare dynamics with the desired PID contribution—the term in braces. We also pick up a back action term $\frac{1}{2}[[X, F_D], F_D]$.

As an example, we can consider a cavity mode a with $L_0 = \sqrt{\gamma_0}a$ and $H_0 = \omega_0 a^*a$ and $F_P = k_P(a + a^*)$, $F_I = k_I(a + a^*)$, and $F_D = k_D(a + a^*)$. Setting $\tilde{a}_t = j_t^{[[Y]]}(a)$ gives

$$d\tilde{a}_t = -\left(\frac{1}{2}\gamma_0 + i\omega_0\right)\tilde{a}_tdt - \sqrt{\gamma_0}dB_t - idW_t, \quad (33)$$

with $dW_t = k_P Y_t dt + k_I\left(\int_0^t Y_s ds\right)dt + k_D dY_t$ corresponding to the PID filtered process as in (32).

Alternatively, we could take $F_P = k_P a^*a$, $F_I = k_I a^*a$, and $F_D = k_D a^*a$, then we find

$$d\tilde{a}_t = -\left(\frac{1}{2}(\gamma_0 - k_D^2) + i\omega_0\right)\tilde{a}_tdt - \sqrt{\gamma_0}dB_t - i\tilde{a}_tdW_t, \quad (34)$$

with W_t again being the PID filtered process in (32). Here the derivative term has altered the damping strength. Otherwise, the three terms enter just into the Hamiltonian $H_t[[Z]]$ leading to a PID filtered version of Y entering as an additional term to the frequency ω_0 .

VI. QUANTUM FEEDBACK NETWORK RULES FOR CONTROLLED MODELS

In this section, we make some rudimentary observations about what the quantum feedback network rules should look like when the various SLH -coefficients of the components are allowed to be adapted processes. We stress that we can only give a sketch of mathematics behind this—the problem of working with a fully rigorous model establishing the self-adjointness of the underlying Hamiltonian and the associated instantaneous feedback limits is well beyond current mathematics in quantum probability. However, leaving aside any pretense at rigor, we can make some reasonable deductions on what to expect.

The derivation of the quantum feedback network rules relies heavily on the Hamiltonian formulation of quantum stochastic calculus derived by Chebotarev,²⁸ for the case of commuting coupling coefficients, and by Gregoratti²⁹ for the general bounded operator case. (The requirement of boundedness was later lifted.³⁰) The unitary stochastic process U_t generated by coefficients $G \sim (S, L, H)$ was shown to be a singular perturbation of the generator of the free shift along the x -axis (with the input being the positive axis and the output line being the negative). We have the space $\mathfrak{h}_0 \otimes \Gamma(L_{\mathbb{C}^n}^2(\mathbb{R}))$ which may be thought of as consisting of vectors $\Psi = (\Psi_m)_{m \geq 0}$ where Ψ_m is in $\mathfrak{h}_0 \otimes \left(\bigotimes_{\text{symm}}^m L_{\mathbb{C}^n}^2(\mathbb{R}) \right)$. That is, we have the functions

$$\begin{aligned} \Psi_m : \mathbb{R}^m \times \{1, \dots, n\} &\mapsto \mathfrak{h}_0, \\ &: (t_1, \dots, t_m, k_1, \dots, k_m) \mapsto \Psi_{k_1, \dots, k_m}(t_1, \dots, t_m), \end{aligned}$$

which are symmetric under interchange of the parameters (t_j, k_j) . We define the operators

$$(b_k(s) \Psi)_{k_1, \dots, k_m}(t_1, \dots, t_m) = \Psi_{k_1, \dots, k_m, k_{m+1}=k}(t_1, \dots, t_m, t_{m+1}=s),$$

where appropriate along with the one-sided annihilators $b_i(0^\pm)$. Specifically, their Hamiltonian, K is given by

$$-iK\Psi = -i\tilde{K}_0\Psi - \left(\frac{1}{2} \sum_k L_k^* L_k + iH \right) \Psi - \sum_{kj} L_k^* S_{kj} b_j(0^+) \Psi, \quad (35)$$

where

$$\tilde{K}_0 = \left(\int_{-\infty}^0 + \int_0^{+\infty} \right) b_j(x)^* \left(i \frac{\partial}{\partial x} \right) b_j(x) dx.$$

Note that \tilde{K}_0 generates translation down the x -axis and is a second quantisation of the momentum operator. The states Ψ live in the Hilbert space $\mathfrak{h}_0 \otimes \Gamma(L_{\mathbb{C}^n}^2(\mathbb{R}))$ and the domain of K consists of suitably regular vectors which satisfy a supplementary boundary condition

$$b_j(0^-) \Psi = L_j \Psi + \sum_k S_{jk} b_k(0^+) \Psi. \quad (36)$$

The quantum feedback network theory builds on this to consider a graph with separate quantum systems at the vertices (as separate $G_k \sim (S_k, L_k, H_k)$ for each one) and propagating Bose fields along the edges. On each edge, we have a sense of direction of propagation. Some of the edges run between vertices—these are the internal ones—while some extend to infinity. The semi-infinite edges then correspond to either input lines (terminating at a vertex) or an output lines (starting at a vertex). The lines can have arbitrary multiplicity for the number of Bose fields they carry and the vertices can have several incoming and outgoing edges; but the total multiplicity in must equal the total multiplicity out. Feedback reduction then takes place by shortening each of the edges down to zero length (instantaneous feedback limit). At each vertex, we have a boundary condition of the form (36). When we eliminate an edge, we get a reduction of order of the graph and a telescoping of the boundary conditions in a systematic manner. Eliminating all internal edges should result in a Markovian model which is an effective model for the network in the instantaneous propagation limit.

For the case of a controlled flow for a single component $G_t[[Z]] \sim (S_t[[Z]], L_t[[Z]], H_t[[Z]])$ considered in this paper, it is natural to argue that the corresponding Hamiltonian should be

$$-iK_t[[Z]] \Psi = -i\tilde{K}_0 \Psi - \left(\frac{1}{2} \sum_k L_{k,t}^*[[Z]] L_{k,t}[[Z]] + iH_t[[Z]] \right) \Psi - \sum_{kj} L_{k,t}^*[[Z]] S_{kj,t}[[Z]] b_j(0^+) \Psi,$$

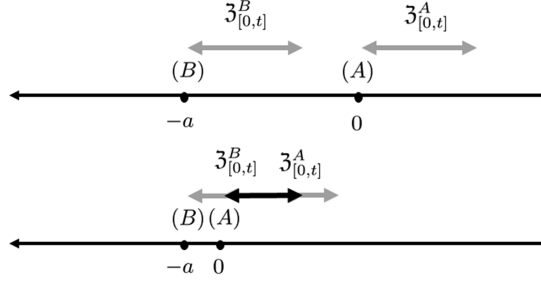


FIG. 3. Systems in series separated by distance a . The control process algebras are sketched for $0 < t < a$ and $t > a$.

with the boundary condition

$$b_j(0^-)\Psi = L_{j,t}[[Z]]\Psi + \sum_k S_{jk,t}[[Z]]b_k(0^+)\Psi.$$

Note that the Hamiltonian and its boundary condition are time dependent. They also depend on the control process Z however this is, in principle, amenable to rigorous formulation. Note that it is essential that we construct the Hamiltonian K in the input picture!

The step up to a non-Markovian network of such is now evident, and in principle the edge elimination should proceed in a similar manner as before leading to the same algebraic form for the rules.

For instance, the series product for systems $G_t^{(A)}[[Z]]$ and $G_t^{(B)}[[Z]]$ should be

$$\begin{aligned} & (S_t^B[[Z]], L_t^B[[Z]], H_t^B[[Z]]) \triangleleft (S_t^A[[Z]], L_t^A[[Z]], H_t^A[[Z]]) \\ &= \left(S_t^B[[Z]] S_t^A[[Z]], L_t^B[[Z]] + S_t^B[[Z]] L_t^A[[Z]], H_t^A[[Z]] + H_t^B[[Z]] + \text{Im} L_t^B[[Z]] S_t^B[[Z]] L_t^A[[Z]] \right). \end{aligned}$$

The cascaded systems may be considered as separated by a distance a before the limit. Taking the speed of propagation to be $c = 1$, we see that the input algebras to A and B at time t are algebras generated by $\{Z_s : 0 \leq s \leq t\}$ and $\{Z_s : -a \leq s \leq t - a\}$, respectively. For t less than a , there is no intersection; however, for $t > a$ they overlap and after $a \rightarrow 0^+$, for t fixed, the two algebras must coincide, see Fig. 3.

We can also consider a beam splitter, as in Fig. 4. While there are no interconnections here, it is worth mentioning that we may want to select different measurements at the two output ports. For instance, at the first output, we may measure the quadrature $Y_1(t) = B_1^{\text{out}}(t) + B_1^{\text{out}*}(t)$ and at the second perform a photon counting measurement $Y_2(t) = \Lambda_{22}(t)$. In this case, we should take $Z_1(t) = B_1(t) + B_1^*(t)$ and $Z_2(t) = \Lambda_{22}(t)$. This may seem strange as we are not measuring the inputs, and the output fields are a linear superposition of the input fields; however, this is the construction

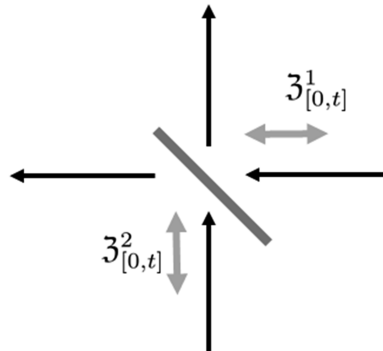


FIG. 4. Inputs into a beam splitter.

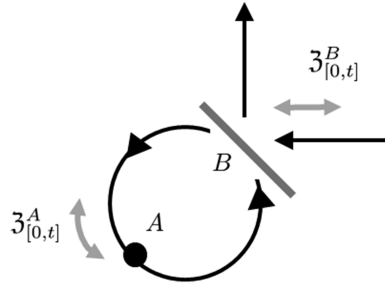


FIG. 5. Algebraic feedback loop.

we have to make if we revert to the input picture. In this case, the two measurement algebras are independent factors of the total noise algebra.

In Fig. 5, we have a simple network consisting of a beam splitter B and a cavity A . The cavity is put into an optical feedback loop using the beam splitter. While the time to propagate around the loop is finite, the model will be non-Markovian—unless the loop field is somehow incorporated into the system. We have sketched the sections of the edges where the input algebras for A and B live, and for t less than the propagation time from B to A they do not intersect. In the instantaneous feedback limit, the loop shrinks to zero. As such, the algebra $\mathfrak{Z}_{[0,t]}^A$ gets pushed back out of the loop and eventually coincides with $\mathfrak{Z}_{[0,t]}^B$.

One therefore reasonably expects the new quantum feedback network rules to be formally the same as for the standard concatenation and feedback reduction rules appearing in Ref. 3, with the modeling proviso that the functional form of the components in terms of Y is decided upon in the output picture, then deduced for the input picture and incorporated into the SLH -coefficients to make them dependent on the various control processes Z .

VII. CONCLUSION

Our discussions have shown that there is an inner consistency within the quantum feedback setup, provided one correctly distinguishes between the input picture and the output picture. Both are invaluable as far as model building and analysis are concerned, but it is necessary to understand the connection between these in order to have an overview of quantum feedback systems.

Unlike the Wiseman paper, where the feedback Hamiltonian is proportional to \dot{Y} (in the output picture), we consider feedback models that are regular. Wiseman's theory was effectively reformulated in the input picture in the paper introducing the series product construction.¹⁸ The present paper opens up a more general theory of quantum feedback where a modulating filter processes the measurement readout and allows us to consider a broader range of feedback scenarios.

A noteworthy feature is that we have not had to go into any specificity about the modulating filter. It could be classical, in which case we just need the input-output relation giving W_t as process adapted to Y . And in particular, we do not have to worry about the mechanism by which the filter works, or how we would model a hybrid classical and quantum system. It could also be quantum in which case its degrees of freedom would have to be taken into account—however these would be independent of the system's and would easily be handled by an augmentation of the results presented in Sec. II absorbing the algebra of the filter observables which of course commutes with \mathfrak{A}_0 and $\mathfrak{N}_{[0,\infty)}$.

The theory set out here provides for great flexibility in modeling and designing quantum control systems. The issue of how exactly we physically realize the filter or how exactly we couple the modulated signal to the SLH -coefficients (that is, engineer a specific Z -dependence on the SLH -coefficients) has not been addressed here. However, the power of the theory is that the specifics are not particularly relevant, and that a consistent systematic approach exists which is not dependent on these details. This now enables us to apply a wide range of standard control engineering methods to quantum open systems.

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